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LETTER TO THE EDITOR

The solutions of the first order matrix differential equation $dU/dr = (A/r - B)U$

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Abstract. The solutions of the matrix differential equation

$$dU/dr = (A/r - B)U$$

for a few specific pairs of 2×2 matrices A and B are obtained. The solutions consist of functions which are related to the Whittaker function.

1. Introduction

The power series as well as asymptotic solutions of the first order matrix differential equation

$$dU/dr = (A/r - B)U, \tag{1}$$

where A and B are arbitrary square matrices and the solution U is also a square matrix, has been investigated by Onley (1972). The Dirac equation (Rose 1961) in the presence of a Coulomb field is separable in spherical coordinates. The two components of the radial part satisfy coupled equations which can be expressed as a matrix equation of the same form as equation (1). The solutions (power series and asymptotic for the Dirac equation) in the representation in which either A or B is diagonal have been investigated in detail by Sud *et al* (1976) and Sud (1976). In § 2 we consider the solution of the matrix equation for a particular pair of matrices A and B for which the solution consists of Whittaker functions. Large numbers of functions are special cases of the Whittaker functions (see Abramowitz and Stegun 1972, p 509). Only a few functions among these special cases satisfy a matrix equation of the form (1). In § 3 we give the specific matrices A and B of equation (1) whose solutions are the special cases of the Whittaker functions (e.g. Bessel functions, Hankel functions, Coulomb functions, Laguerre polynomials etc). The matrix representation of these functions helps in both formal manipulation and computation. It also facilitates the evaluation of an integral over the products of such functions by expressing it in terms of the matrix generalisation of the gamma function. Suppose we have (2×2) matrix equations of the same form as equation (1) distinguished by the suffix $i = 1, 2, \ldots, n$

$$dU_i(k_ir)/dr = (A_i/r - B_i)U_i(k_ir). \tag{2}$$

The direct product of such solutions is written as

$$W(r) = U_1(k_1r)\otimes U_2(k_2r)\otimes\ldots\otimes U_n(k_nr) \tag{3}$$

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and is a $2^n \times 2^n$ matrix. Furthermore, $W(A, B: r)$ is a solution of an equation of the same form as equation (1) with

$$A = A_n \otimes I_{2n-2} + I_2 \otimes A_{n-1} \otimes I_{2n-4} + \ldots + I_{2n-2} \otimes A_1$$

$$B = k_n B \otimes I_{2n-2} + k_{n-1} I_2 \otimes B \otimes I_{2n-4} + \ldots + k_1 I_{2n-2} \otimes B$$

where I_n is an $n \times n$ matrix. Thus we can find a single matrix series for the product of functions by using re-defined matrices A and B and the general solution of equation (1). Onley (1972) and Sud *et al* (1976) have defined an integral over the product of the solutions as follows:

$$\Gamma(A+1, B) = \int_{(0)}^{\infty} W(A, B: r)\,\mathrm{d}r \tag{4}$$

where the lower limit (0) in the above integral implies that the singularities at this limit have been removed. $\Gamma(A+1, B)$ is a $2^n \times 2^n$ matrix and is known as the matrix gamma function. (For explicit expression in terms of matrices A and B see Sud *et al* (1976).)

Furthermore, Wright *et al* (1977) have found that the matrix gamma function satisfies an equation of the form,

$$\partial\Gamma/\partial k_i = T_i \Gamma \qquad i = 1, 2, \ldots, n \tag{5}$$

where T_i is a constant matrix and a function of A and B. The above equation can be integrated numerically to evaluate the variation of Γ with respect to the parameter k_i. One only needs the matrices A and B in order to use equations (3), (4) and (5) as shown in § 3.

2. The solution of the matrix equation

We shall investigate the differential recurrence relations for the Whittaker function (Abramowitz and Stegun 1972) and from there extract the pair of matrices A and B. The differential recurrence relations satisfied by the Whittaker function $M_{q,\mu}(z)$ are given as follows:

$$\frac{\mathrm{d}M_{q,\mu-\frac{1}{2}}(z)}{\mathrm{d}z} = \left(\frac{\mu}{z} - \frac{q}{2\mu}\right) M_{q,\mu-\frac{1}{2}}(z) + \frac{\mu^2 - q^2}{4\mu^2(2\mu+1)} M_{q,\mu+\frac{1}{2}}(z) \tag{6}$$

$$\frac{\mathrm{d}M_{q,\mu+\frac{1}{2}}(z)}{\mathrm{d}z} = \left(-\frac{\mu}{z} + \frac{q}{2\mu}\right) M_{q,\mu+\frac{1}{2}}(z) + (2\mu+1) M_{q,\mu-\frac{1}{2}}(z). \tag{7}$$

These can be combined and expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} M_{q,\mu-\frac{1}{2}}(z) \\ M_{q,\mu+\frac{1}{2}}(z) \end{pmatrix} = \left[\begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}\frac{1}{z} - \begin{pmatrix} q/2\mu & -(\mu^2-q^2)/4\mu^2(2\mu+1) \\ -(2\mu+1) & -q/2\mu \end{pmatrix}\right]\begin{pmatrix} M_{q,\mu-\frac{1}{2}}(z) \\ M_{q,\mu+\frac{1}{2}}(z) \end{pmatrix}. \tag{8}$$

The matrices A and B can be identified from (8) as

$$A = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}; \qquad B = \begin{pmatrix} q/2\mu & -(\mu^2-q^2)/4\mu^2(2\mu+1) \\ -(2\mu+1) & -q/2\mu \end{pmatrix}$$

and the solution is given by

$$U_0^{(A)}(A, B: z) = \begin{pmatrix} M_{q,\mu-\frac{1}{2}}(z) & \dfrac{\mu+q}{2\mu(2\mu-1)} M_{q,-\mu+\frac{1}{2}}(z) \\ \dfrac{q-\mu}{2\mu(2\mu+1)} M_{q,\mu+\frac{1}{2}}(z) & M_{q,-\mu-\frac{1}{2}}(z) \end{pmatrix}. \tag{9}$$

The superscript A (or B) indicates that A (or B) is a diagonal matrix and subscript 0 denotes a power series solution. The first column of the matrix solution $U_0^{(A)}(A, B: z)$ is regular at the origin whereas the second column represents the irregular solution. We can have a B-diagonal representation by a similarity transformation of the equation (8). In the B-diagonal representation the matrices are,

$$A = \begin{pmatrix} -(q+\frac{1}{2}) & q+\mu+\frac{1}{2} \\ \mu-q-\frac{1}{2} & -(q+\frac{1}{2}) \end{pmatrix}; \qquad B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

and the solution $U_0^{(B)}(A, B: z)$ is given by:

$$U_0^{(B)}(A, B: z) = z^{-1/2} \begin{pmatrix} M_{q,\mu}(x) & \dfrac{q+\frac{1}{2}+\mu}{q+\frac{1}{2}-\mu} M_{q,-\mu}(z) \\ M_{q+1,\mu}(z) & M_{q+1,-\mu}(z) \end{pmatrix}. \tag{10}$$

3. Special cases

In this section we shall seek the solution of the matrix equation (1) for the particular matrices A and B. In particular we find the solutions which consist of Bessel functions, Hankel functions, Laguerre polynomials, Coulomb wavefunctions and Cunningham functions.

(*a*) For the matrices

$$A = \begin{pmatrix} -n & 0 \\ 0 & n-1 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix}. \tag{11}$$

The A-diagonal solution of the matrix equation can be identified as

$$U_0^{(A)}(A, B: r) = \begin{pmatrix} N_n(wr) & J_n(wr) \\ N_{n-1}(wr) & J_{n-1}(wr) \end{pmatrix} \tag{12}$$

where $J_n(wr)$ is the Bessel function and $N_n(wr)$ is the Neumann function. An asymptotic solution can be obtained in the A-diagonal representation as

$$U_0^{(A)}(A, B: r) = U_\infty^{(A)}(A, B: r) T \tag{13}$$

where the asymptotic solution $U_\infty^{(A)}(A, B: r)$ and the constant matrix T are given by

$$U_\infty^{(A)}(A, B: r) = \begin{pmatrix} H_n^{(1)}(wr) & H_n^{(2)}(wr) \\ H_{n-1}^{(1)}(wr) & H_{n-1}^{(2)}(wr) \end{pmatrix}$$

and

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\mathrm{i} & -\frac{1}{2}\mathrm{i} \end{pmatrix} \tag{14}$$

where $H_n^{(1)}(wr)$ and $H_n^{(2)}(wr)$ are Hankel functions of the first and second kind respectively.

(*b*) For the pair of matrices

$$A=\begin{pmatrix}-n-1 & 0\\ 0 & n-1\end{pmatrix}; \qquad B=\begin{pmatrix}0 & -w\\ w & 0\end{pmatrix}. \tag{15}$$

The $U_0^{(A)}(A, B\colon r)$ is given by

$$U_0^{(A)}(A, B\colon r)=\begin{pmatrix}n_n(wr) & j_n(wr)\\ n_{n-1}(wr) & j_{n-1}(wr)\end{pmatrix} \tag{16}$$

where $j_n(wr)$ and $n_n(wr)$ are spherical Bessel and spherical Neumann functions respectively. An asymptotic solution can be obtained by using equation (13). The asymptotic solution $U_\infty^{(A)}(A, B\colon r)$ and constant matrix T are given by

$$U_\infty^{(A)}(A, B\colon r)=\begin{pmatrix}h_n^{(1)}(wr) & h_n^{(2)}(wr)\\ h_{n-1}^{(1)}(wr) & h_{n-1}^{(2)}(wr)\end{pmatrix}$$

and

$$T=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2}\mathrm{i} & -\frac{1}{2}\mathrm{i}\end{pmatrix} \tag{17}$$

where $h_n^{(1)}(wr)$ and $h_n^{(2)}(wr)$ are spherical Hankel functions of the first and second kind respectively.

(*c*) For the matrices

$$A=\begin{pmatrix}-L & 0\\ 0 & L\end{pmatrix}; \qquad B=\begin{pmatrix}-\eta/L & [(L^2+\eta^2)/L^2]^{-\frac{1}{2}}\\ -[(L^2+\eta^2)/L^2]^{\frac{1}{2}} & \eta/L\end{pmatrix}. \tag{18}$$

The A-diagonal matrix solution of an equation with the above pair of matrices is given by

$$U_0^{(A)}(A, B\colon \rho)=\begin{pmatrix}F_L(\eta,\rho) & G_L(\eta,\rho)\\ F_{L-1}(\eta,\rho) & G_{L-1}(\eta,\rho)\end{pmatrix} \tag{19}$$

where $F_L(\eta,\rho)$ and $G_L(\eta,\rho)$ are regular and irregular Coulomb wavefunctions respectively.

(*d*) For the pair of matrices

$$A=\begin{pmatrix}-(n+1+\frac{1}{2}\alpha) & n+1+\alpha\\ -n-1 & n+1+\frac{1}{2}\alpha\end{pmatrix}; \qquad B=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

The B-diagonal solution for the above pair of matrices is given by

$$U_0^{(B)}(A, B\colon z)=\begin{pmatrix}z^{\frac{1}{2}\alpha}L_n^\alpha(z) & (n+1)z^{-\frac{1}{2}\alpha}L_{n+\alpha}^{-\alpha}(z)\\ z^{\frac{1}{2}\alpha}(n+1)L_{n+1}^\alpha(z) & z^{-\frac{1}{2}\alpha}L_{n+\alpha+1}^{-\alpha}(z)\end{pmatrix}$$

where $L_n^\alpha(z)$ is the associated Laguerre polynomial.

(*e*) For matrices

$$A=\begin{pmatrix}\frac{1}{2}m+n & \frac{1}{2}m-n\\ \frac{1}{2}m+n & \frac{1}{2}m-n\end{pmatrix}; \qquad B=\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}.$$

The B-diagonal solution with the above pair of matrices is given by:

$$U_0^{(B)}(A, B: z) = \begin{pmatrix} e^{\frac{3}{2}z} W_{n,m}(z) & e^{-\frac{3}{2}z} W_{-n-1,m}(-z) \\ e^{\frac{3}{2}z} W_{n-1,m}(z) & e^{-\frac{3}{2}z} W_{-n,m}(-z) \end{pmatrix}$$

where $W_{n,m}$ is the Cunningham function.

In conclusion we have given the matrix solutions of the matrix equation which consist of functions of great use in various problems of mathematical physics.

References

Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (Washington: National Bureau of Standards) p 509

Onley D S 1972 *Nuclear Structure Studies using Electron Scattering and Photoreaction* (Sendai: Tohoku University)

Rose M E 1961 *Relativistic Electron Theory* (New York: Wiley)

Sud K K 1976 *PhD Thesis* Ohio University

Sud K K, Wright L E and Onley D S 1976 *J. Math. Phys.* **17** 2175

Wright L E, Onley D S and Soto Vargas C W 1977 *J. Phys. A: Math. Gen.* **10** L53